1·0   2·8   2·5

5·0   3·15   2·2
5·6
6·1   3·5

1·1   4·0   2·0
4·5

1·8

1·25   1·4   1·6

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FOREIGN TECHNOLOGY DIVISION

NET-CHARACTERISTIC METHOD FOR THE NUMERICAL
SOLUTION OF PROBLEMS OF GAS DYNAMICS

by

K.M. Magomedov

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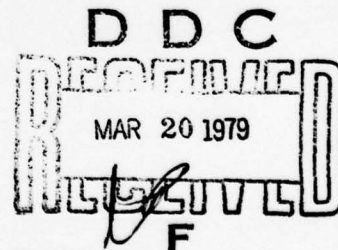
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SOLUTION OF PROBLEMS OF GAS DYNAMICS

By: K. M. Magomedov

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Block	Italic	Transliteration	Block	Italic	Transliteration
А а	*А а*	A, a	Р р	*Р р*	R, r
Б б	*Б б*	B, b	С с	*С с*	S, s
В в	*В в*	V, v	Т т	*Т т*	T, t
Г г	*Г г*	G, g	У у	*У у*	U, u
Д д	*Д д*	D, d	Ф ф	*Ф ф*	F, f
Е е	*Е е*	Ye, ye; E, e*	Х х	*Х х*	Kh, kh
Ж ж	*Ж ж*	Zh, zh	Ц ц	*Ц ц*	Ts, ts
З з	*З з*	Z, z	Ч ч	*Ч ч*	Ch, ch
И и	*И и*	I, i	Ш ш	*Ш ш*	Sh, sh
Й й	*Й й*	Y, y	Щ щ	*Щ щ*	Shch, shch
К к	*К к*	K, k	Ъ ъ	*Ъ ъ*	"
Л л	*Л л*	L, l	Ы ы	*Ы ы*	Y, y
М м	*М м*	M, m	Ь ь	*Ь ь*	'
Н н	*Н н*	N, n	Э э	*Э э*	E, e
О о	*О о*	O, o	Ю ю	*Ю ю*	Yu, yu
П п	*П п*	P, p	Я я	*Я я*	Ya, ya

*ye initially, after vowels, and after ъ, ь; e elsewhere.
When written as ё in Russian, transliterate as yё or ё.

RUSSIAN AND ENGLISH TRIGONOMETRIC FUNCTIONS

Russian	English	Russian	English	Russian	English
sin	sin	sh	sinh	arc sh	$\sinh^{-1}$
cos	cos	ch	cosh	arc ch	$\cosh^{-1}$
tg	tan	th	tanh	arc th	$\tanh^{-1}$
ctg	cot	cth	coth	arc cth	$\coth^{-1}$
sec	sec	sch	sech	arc sch	sech^{-1}
cosec	csc	csch	csch	arc csch	csch^{-1}

Russian	English
rot	curl
lg	log

i

0951

NET-CHARACTERISTIC METHOD FOR THE NUMERICAL SOLUTION OF PROBLEMS OF GAS DYNAMICS

K. M. Magomedov

Various numerical methods of solving multidimensional problems for equations in partial derivatives have undergone intensive development and introduction in recent years. Different versions of the net method proposed and studied in the works of K. I. Babenko, V. V. Rusanov, N. N. Yanenko, S. K. Godunov, etc., the Dorodnitsyn-Belotserkovskiy method of integral relationships, the straight line method developed by G. F. Telenin et coworkers, and difference systems using the characteristic properties of equations [1-9] are being used successfully for solving specific problems of gas dynamics.

Numerical methods using characteristics have definite advantages over ordinary finite-difference methods: in particular, the consideration of the physical nature of the problem, the small variation in certain complexes from the unknown functions along the characteristics, and the possibility of predicting the time of the origination of discontinuities, e.g., "suspended" shocks. These advantages become especially clear when solving problems with two independent variables. However, the effectiveness of the characteristic method considerably decreases with the increase in the dimensionality of space, and virtually vanishes when there are more than three independent variables.

Thus, it becomes desirable to construct difference systems which retain the main advantages of the characteristic method, but have the convenience of the net method which results from the fixed net and the simplicity of the calculation formulae.

Obviously, the name "characteristic method" should only be used for numerical systems in which characteristic curves or surfaces are set up during the calculation process. Then numerical systems in which the characteristic relationships (compatibility conditions) are only used to derive the difference equations at fixed nodes can be considered to be a version of the net method. This is even more expedient because it is precisely for this type of system that the

problem of stability arises, even when the Courant-Freidrichs-Levi
(CFL) condition is satisfied, i.e., the range cf the dependence for
the differential equations lies inside the range of the dependence of
the difference equations [1C-12].

There is great flexibility in the selection of numerical methods
based on characteristic relationships in multidimensional problems.
Report [17] gives a rather detailed survey cf the specific methods of
this type which have been developed for gas-dynamic equations. But,
obviously, it makes sense to have a sufficiently general and simple
method of constructing explicit difference systems for hyperbolic
multidimensional equations similar to the Ccurant, Isacson, Rees
method [13] or the "running calculation" method (see [15]) for
systems with two independent variables.

The main idea of the net-characteristic method proposed in [18]
is that difference approximations of the compatibility conditions and
linear or quadratic interpolation by the nodes of a preassigned fixed
network are used for each elementary cell. On the other hand, the
method can be considered to be the ordinary method of nets with a
specific rule for the difference approximation cf the initial
equations. This is the viewpoint from which the net-characteristic
method of constructing difference systems is considered below.

§1 gives the main concept of the net-characteristic method and some forms of compact notation of difference equations of the initial matrices using fundamental matrices. §2 investigates the local stability of the corresponding difference systems. §3 shows that the net-characteristic method can be used to construct divergent difference systems. First-order systems are the simplest, being monotonic and flexible.

A number of problems of gas dynamics have been solved by the net-characteristic method:

- supersonic steady flow about both convex and concave sharp and bluff bodies at an angle of attack;

- the direct problem of calculating flow in a nozzle;

- three-dimensional flow about segmented conical bodies, etc.

The last problems were solved by the determination method which, in the most interesting cases, means solving equations in partial derivatives with four independent variables. §4 gives some results of the calculations as an illustration.

§1. Net-Characteristic Method of Constructing Difference Systems for Quasilinear Hyperbolic Equations

First we will consider a system of first-order hyperbolic quasilinear equations in partial derivatives with two independent variables t, x

$$U_t + A U_x = f \tag{I.I}$$

where the matrix A and the vector column of the right sides f can be functions of t, x and vector function u of unknown variables with N components.

Let μ_i and ω_i ($i = 1, 2, \ldots, N$) be proper real numbers and the independent latent vectors of transposed matrix A' corresponding to them. Multiplying (1.1) in scalar form by ω_i and considering the equation $A'\omega_i = \mu_i \omega_i$, we will reduce the system to the characteristic (normal) form

$$\omega_i u_{t_i} = \omega_i f \quad (u_{t_i} = u_t + \mu_i u_x, \ i = 1, 2, \ldots, N) \tag{I.2}$$

Suppose that we know solution (1.1) in layer $t = t_0 = $ const. We will

find the value of $U_H = U(H)$ at point H of a close layer $t = t_0 + \tau$.
Approximating (1.2) by difference equations at $(0 \leqslant \nu \leqslant 1)$, we will
have

$$\omega_i(u_H - u_i) = \tau[\nu\omega_i f_i + (1-\nu)\omega_i f_H] + O[(1-2\nu)\tau^2 + \tau^3] \qquad (I.3)$$

Here the subscript i designates the values of the functions at the
points of contact of the characteristics drawn from point H with line
$t = t_0$. We will point out that it is necessary to compute ω_i from the
mean parameters of points H and i in order to obtain the second-order
approximation for quasilinear equations at $\nu = {}^1/_2$ in (1.3).

Based on the linear independence of vectors U_H, system (1.3) has
a single solution relative to

$$u_H = F(t, x, u(t_0, x), u_H, \nu) + O[(1-2\nu)\tau^2 + \tau^3]$$

We will consider the fixed network $t = n\tau (n = 0, 1, 2, \ldots)$,
$x = mh (m = 0, \pm 1, \pm 2, \ldots)$ and we will designate the values of the functions
at the nodes of the network by u_m^n. If the relationship of the
parameters with subscript i to the known values of u_m^n in layers $t = t_n = n\tau$ is indicated, we can find a difference system with a fixed
network, or the net method. When using three points for linear or
quadratic interpolation, we will have, respectively $(\sigma = \tau/h)$:

$$u_i = u_m^n - \delta \mu_i \frac{u_{m+1}^n - u_{m-1}^n}{2} + \delta |\mu_i| \frac{u_{m+1}^n + u_{m-1}^n - 2 u_m^n}{2} + O(t_i^2) \quad (1.4.I)$$

$$u_i = u_m^n - \delta \mu_i \frac{u_{m+1}^n - u_{m-1}^n}{2} + \delta^2 \mu_i^2 \frac{u_{m+1}^n + u_{m-1}^n - 2 u_m^n}{2} + O(h^3) \quad (1.4.II)$$

Relationships (1.3) and (1.4) completely define the difference system, but it is expedient to consider its compact form. We will introduce the following designations

$$\Delta_m u = \tfrac{1}{2}(u_{m+1}^n - u_{m-1}^n)$$
$$\Delta_m^2 u = \tfrac{1}{2}(u_{m+1}^n + u_{m-1}^n - 2 u_m^n) \qquad \Omega = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \end{pmatrix}$$

Obviously, matrix Ω, whose lines are the linearily independent vectors ω_i, is not confluent and has the inverse matrix Ω^{-1}. Writing (1.3) in matrix form and using the result of (1.4),

$$\Omega u_i = \Omega u_m^n - \delta \Lambda \Omega \Delta_m u + \delta^2 \Lambda^\gamma \Omega \Delta_m^2 u$$

where $\gamma = 1$ (2) in the case of linear (quadratic) interpolation, Λ . Λ^γ are diagonal matrices with elements $\{\mu_1, \mu_2, \ldots, \mu_n\}$ and $\{|\mu_1|^\gamma, |\mu_2|^\gamma, \ldots, |\mu_n|^\gamma\}$, we can find

$$u_m^{n+1} = u_m^n - \delta \Omega^{-1} \Lambda \Omega \Delta_m u + \delta^2 \Omega^{-1} \Lambda^2 \Omega \Delta_m^2 u + \tag{1.5}$$
$$+ \tau \{ \nu f_m^n + (1-\nu) f_m^{n+1} - \nu \delta \Omega^{-1} \Lambda \Omega \Delta_m f \} + O[(1-2\nu) \tau^2, \tau^3]$$

The compatibility conditions (1.2) in a fixed system of nodes using
interpolation formula (1.4) were used to derive difference systems
(1.5). But the difference formulae can be written immediately if we
know the fundamental matrix Ω for A, i.e., the matrix which reduces A
to diagonal form. We will point out the two most important systems in
(1.5): first-order system I with linear interpolation ($\gamma = 1$, $\nu = 0$,
then $A = \Omega^{-1} \Lambda \Omega$ and f can be computed at point $t = n\tau$, $x = mh$
without disturbing the order of the approximation), and system II
with quadratic interpolation ($\gamma = 2$, $\nu = 1/2$).

We will now consider the multidimensional case

$$u_t + A_1 u_{x_1} + A_2 u_{x_2} + \cdots = f \tag{1.6}$$

We will limit ourselves to the terms described in (1.6) when deriving
difference formulae of the type in (1.5), since the extension of the
method to a large number of variables is done formally. We will
assume that matrices A_1, A_2, ... have only proper real values. Let μ_i^α
and ω_i^α ($i = 1, 2, \ldots, N$) be proper numbers and the independent
latent vectors of matrix A^1 ($\alpha = 1, 2$) corresponding to them. Then

equations (1.6) can be reduced to one of the following characteristic forms

$$
\begin{aligned}
&\omega_i^1 u_{t_i} = \omega_i^1 (f - A_2 u_{x_2}) \\
&\quad (u_{t_i} = u_t + \mu_i^1 u_{x_1}, \quad i = 1, 2, \ldots, N) \\
&\omega_j^2 u_{t_j} = \omega_j^2 (f - A_1 u_{x_1}) \\
&\quad (u_{t_j} = u_t + \mu_j^2 u_{x_2}, \quad j = 1, 2, \ldots, N)
\end{aligned}
\tag{1.7}
$$

Here u_{t_i}, u_{t_j} designate the derivatives along the two-dimensional characteristics on the surfaces x_2 = const and x_1 = const, respectively, or, in other words, along the lines of intersection of the characteristic and coordinate surfaces. The directions of these lines are determined by the vectors $\{1, \mu_i^1, 0\}$ and $\{1, 0, \mu_j^2\}$.

We will now consider 3N equations (1.6)-(1.7) jointly. Obviously, the number of independent equations among them is equal to N. This fact can be used to construct convenient explicit difference systems for multidimensional equations, in particular, those not requiring the approximation of the partial derivatives for x_1 and x_2. Replacing equations (1.6)-(1.7) by the finite-difference relationships along the corresponding lines at $t = t_0 + (1 - \nu)\tau$ and multiplying (1.7) by Ω_α^{-1} (Ω_α is a matrix whose lines are vectors (ω_i^α ($\alpha = 1, 2$), we will have:

$$\frac{u_H - u_0}{\tau} = (1-\nu)\left[f - A_1 u_{x_1} - A_2 u_{x_2} \right]_H + \nu\left[f - A_1 u_{x_1} - A_2 u_{x_2} \right]_0 + \cdots$$

$$\frac{u_H - \Omega_1^{-1}(\Omega_1 u_i)}{\tau} = (1-\nu)\left[f - A_2 u_{x_2} \right]_H + \nu\left[f - A_2 u_{x_2} \right]_0 + \cdots$$

$$\frac{u_H - \Omega_2^{-1}(\Omega_2 u_i)}{\tau} = (1-\nu)\left[f - A_1 u_{x_1} \right]_H + \nu\left[f - A_1 u_{x_1} \right]_0 + \cdots$$

Subtracting the first equation from the sum of the last equations and using interpolation formulae (1.4) to find $\Omega_1 U$ and $\Omega_2 u_i$, we will obtain the unknown formula

$$\begin{aligned}
u_{m\ell}^{n+1} = u_{m\ell}^n &- \sigma_1 \Omega_1^{-1} \Lambda \Omega_1 \Delta_m u + \sigma_1^\gamma \Omega_1^{-1} \Lambda_1^\gamma \Omega \Delta_m^2 u - \\
&- \sigma_2 \Omega_2^{-1} \Lambda_2 \Omega_2 \Delta_\ell u + \sigma_2^\gamma \Omega_2^{-1} \Lambda_2^\gamma \Omega \Delta_\ell u + \\
&+ \tau\left\{ \nu f_{m\ell}^m + (1-\nu) f_{m\ell}^{n+1} - \nu\left[\sigma_1 A_1 \Delta_m f + \sigma_2 A_2 \Delta_\ell f \right] + \right. \\
&+ \nu\tau^2\left[A_1 (A_2 u_{x_2})_{x_1} + A_2 (A_1 u_{x_1})_{x_2} + O\left[(1-2\nu)\tau^2 + \tau^3 \right] \right. \qquad (I.8)
\end{aligned}$$

Here $\sigma_i = \tau/h_i$ $(i=1,2)$, $x_1 = mh_1$, $x_2 = \ell h_2$. When deriving (1.8), we consider that it **suffices to take the coefficient** at $\nu\tau^2$ with accuracy down to the terms on the order of $O(\tau)$. Furthermore, this term must only be considered for second-order system II ($\gamma = 2$, $\nu = 1/2$), whereupon it is necessary to use one more point in layer $t = n\tau$ to calculate the mixed derivatives.

It should also be pointed out that for the first-order system ($\nu = 0$) - system I - analogously to (1.5) and (1.8), we can immediately write the formulae for any number of three-dimensional coordinates, as long as matrix A_α is reduced to the diagonal form

$$A_\alpha = \Omega_\alpha^{-1} \Lambda_\alpha \Omega_\alpha \quad (\alpha = 1, 2, \ldots)$$

§2. Stability of Net-Characteristic Method

We will consider the differential equations (1.1) and the difference equations approximating them (1.5) at $A = $ const, $f \equiv 0$ and periodic boundary conditions. Then the stability and, therefore, the convergence of the difference systems, which follows from it according to the to the equivalence theorem, can be obtained by analyzing solution (1.5) in the form [14]

$$U_m^n = \upsilon^n e^{im\varphi}, \quad \upsilon^n = G^n \upsilon^0 \ (\varphi = \kappa h, \ \kappa = 0, \pm 1, \pm 2, \ldots)$$
$$G = G(\tau; \varphi) = E - \sigma^\nu \Omega^{-1} \Lambda^\nu \Omega (1 - \cos\varphi) - i\sigma \Omega^{-1} \Lambda \Omega \sin\varphi$$

Neumann's necessary condition of stability

$$\max_i |\lambda_i| = \max_i \left\{ [1 - \sigma^\nu |\mu_i|^\nu (1 - \cos\varphi)]^2 + \sigma^2 \mu_i^2 \sin^2\varphi \right\}^{1/2} \leq 1$$

where λ_i is the proper value of the transitional matrix G, will be
satisfied at $\sigma|\mu_i| \leq 1$ (i = 1, 2, ..., N), i.e., when the CPL
condition is **satisfied**. This same condition is also sufficient, since
limitation G^n, uniform with respect to τ, follows from the
relationship

$$G^{\cdot} = \Omega^{-1} \begin{pmatrix} \lambda_1^n & & 0 \\ & \ddots & \\ 0 & & \lambda_n^n \end{pmatrix} \Omega$$

Thus, systems I and II of the net-characteristic method are
stable in the case of two independent variables. We will point out
that for the first-order system - system I - the stability of the
Cauchy difference problem can be proven with less stringent
limitations. The proof of the convergence of a system similar to
(1.5) is given in [13] for quasilinear system (1.1). At $\gamma = 1$,
relationship (1.5) can be written as

$$U_m^{n+1} = T_0 u_m^n + T_1 u_{m+1}^n + T_{-1} u_{m-1}^n \quad (T_0 + T_1 + T_{-1} = E) \qquad (2.1)$$

$$T_0 = E - \sigma \Omega^{-1} \Lambda^2 \Omega, \quad T_1 = \frac{\sigma}{2} [\Omega^{-1}(\Lambda^2 - \Lambda)\Omega], \quad T_{-1} = \frac{\sigma}{2} [\Omega^{-1}(\Lambda^2 - \Lambda)\Omega]$$

Since $T_i \, (i = 0, \pm 1)$ are matrices nonnegatively determined at
$\sigma \|\Lambda\| \leq 1$, (2.1) is a positive Friedrichs system. Book [15]
considers certain similar systems; in particular, the running
calculation system, which is similar to system (1.6) or system [13].

We will now consider the case of three independent variables
(1.6). At $\nu = 0$ ($\gamma = 1$), for (1.8) the transitional matrix

$$G = G(\tau, \varphi_1, \varphi_2) \quad (\varphi_\alpha = K_\alpha h_\alpha, \ K_\alpha = 0, \pm 1, \pm 2, \ldots, \alpha = 1.2)$$

has the form:

$$G = E - \sigma_1(1 - \cos\varphi_1)\tilde{A}_1 - i\sigma_1 \sin\varphi_1 A_1 -$$

$$- \sigma_2(1 - \cos\varphi_2)\tilde{A}_2 - i\sigma_2 \sin\varphi_2 A_2$$

$$A_\alpha = \mathcal{Q}_\alpha^{-1} \Lambda_\alpha \mathcal{Q}_\alpha, \quad \tilde{A}_\alpha = \mathcal{Q}_\alpha^{-1} \Lambda_\alpha^2 \mathcal{Q}_\alpha \quad (\alpha = 1.2)$$

We will find the Neumann stability condition with the assumption of
the symmetry of matrix A_α. Let $z = x + iy$ be the latent vector G.
Considering equation $(Gz, \bar{z}) = \lambda |z|^2$ and the symmetry of matrix A_1, we
can find

$$|\lambda|^2 \leq [1 - \sigma_1 Q_1(1 - \cos\varphi_1) - \sigma_2 Q_2(1 - \cos\varphi_2)]^2 + [\sigma_1 |P_1 \sin\varphi_1| + \sigma_2 |P_2 \sin\varphi_2|]^2 \quad (2.2)$$

$$Q_\alpha = \frac{(\tilde{A}_\alpha x, x) + (\tilde{A}_\alpha y, y)}{|z|^2} \leq |\mu^\alpha|_{max} = \max_i |\mu_i^\alpha|$$

$$P_\alpha = \frac{(A_\alpha x, x) + (A_\alpha y, y)}{|z|^2} \leq Q_\alpha, \quad |P_\alpha| \leq Q_\alpha \quad (\alpha = 1.2)$$

It is evident from inequality (2.2) that the Neumann stability
condition $|\lambda|^2 \leq 1$ will be valid at $\sigma_1 Q_1 + \sigma_2 Q_2 \leq \max_{i,j}(\sigma_1|\mu_i^1| + \sigma_2|\mu_j^2|) \leq 1$, i.e.,
when the CFL condition is satisfied. The symmetry of A was used in
deriving this equation. But the consideration of individual examples
of asymmetrical matrices makes it possible to hypothesize that the
Neumann condition is satisfied for system I (1.8) when the CFL
condition is satisfied, and in general.

In the case of one scalar equation (1.6) $(N = 1)$, stability
condition (2.2) assumes the form

$$|\lambda|^2 \leq \left[1 - \sigma_1 A_1 (1 - \cos\varphi_1) - \sigma_2 A_2 (1 - \cos\varphi_2)\right]^2 +$$
$$+ \left[\sigma_1 A_1 |\sin\varphi_1| + \sigma_2 A_2 |\sin\varphi_2|\right]^2 \leq 1 \qquad (2.3)$$

This relationship, which is a necessary and sufficient condition for
stability in the case of constant coefficients, is equivalent to
condition (2.2), at least, for symmetrical matrices. Therefore, the

main properties of the difference systems of the method proposed
above can be revealed in the simplest example

$$U_t + A_1 U_{x_1} + A_2 U_{x_2} = f \quad (A_\alpha \geqslant 0, \; \alpha = 1.2) \tag{2.4}$$

Adding a new point $(m - 1, \ell - 1)$, selected with consideration of the
direction of the characteristic, to the five symmetrical points in
layer $t = t_n$, the difference systems of net-characteristic method
(1.8) can be written as follows for equation (2.4):

$$U_{m\ell}^{n+1} = (1 - a_1^\gamma - a_2^\gamma + 2\nu a_1 a_2) U_{m\ell}^n + 2\nu a_1 a_2 \; U_{m-1, \ell-1}^n -$$
$$- \frac{a_1}{2}(1 - a_1^{\gamma-1}) U_{m+1, \ell}^n + \frac{a_1}{2}(1 + a_1^{\gamma-1}) U_{m-1, \ell}^n - \frac{a_2}{2}(1 - a_2^{\gamma-1}) U_{m, \ell+1}^n +$$
$$+ \frac{a_2}{2}(1 + a_2^{\gamma-1}) U_{m, \ell-1}^n + \tau [\nu(f_1 + f_2 - f_0) + (1-\nu) f_{m\ell}^n] +$$
$$+ 0 [(1 - 2\nu)\tau^2 + \tau^3 + h_1^3 + h_2^3] \tag{2.5}$$
$$(a_1 = \frac{A_1 \tau}{h_1}, \; a_2 = \frac{A_2 \tau}{h_2})$$

At $\nu = 0$, $\gamma = 1$ we have system I. It is easy to check the
stability of this system when the CFL condition is satisfied by the
subscript of the difference system, since the system is positive:

$$I = |1 - a_1 - a_2| + a_1 + a_2 + c\tau \sim 1 + c\tau$$
$$a_1 + a_2 \leqslant 1, \quad c = \max_{m\ell} |f_{m\ell}^n|$$

We will point out that this system is monotonic, i.e., the monotonic

profiles are also transformed into monotonic profiles at $\ell \equiv 0$.

Using the Fourier method, i.e., relationships like (2.3), we can study the difference systems obtained from (2.5). We will point out some of them, omitting the calculations.

During quadratic interpolation ($\gamma = 2$), the system turns out to be stable when the CFL condition is only satisfied for $\nu = 1/2$ (system II). The system is stable for the rest of the values of ν when $a_i^2 = 0(\tau)$, i.e., when the time interval is considerably smaller than the spatial coordinate interval. Then $\max|\lambda| = 1 + \alpha_1^2 + \alpha_2^2$, and the growth of the initial error ε_0 during the calculation is characterized by the value $\varepsilon_0 (1 + \alpha_1^2 + \alpha_2^2)^{\star k}$.

The use of additional nodes on plane $t = n\tau$ lowers stability. For example, the use of linear interpolation with the satisfaction of the CFL condition when using yet another point ($m - 2, \ell$) makes the system generally unstable. Systems which are only stable at $a_i^2 = 0(\tau)$ are obtained during high-order interpolation using new points.

Systems can exist which are stable when the CFL condition is satisfied in one direction, e.g., $\alpha_t < 1$, and $a_\lambda^2 = 0(\tau)$ is required for the other variable. Specifically, this is the system previously proposed by the author of [8]. During the practical calculation using

this system, the interval of variable x_2 was selected on the basis of the nature of the solution cf the problem sc that $a_2^2 \sim 10^{-4}$. This results in a 12o/o increase in the initial error ε_0 after 1000 intervals on t, which is completely permissible. This example indicates that these systers can be used when $a_1^2 \ll 1$.

The nature of the suppression (or propagation) of the errors arising during the calculation is an important property of any difference system, as well as the behavior cf the net functions in the region of an irregular solution. Relaticnships (2.5) and the practical calculations show that system I mcnotcaically decreases the error (this is logical, since this system is positive), while system II also decreases the amplitude cf the errcr, but with fluctuations cf different signs. A similar situation occurs for more complex equations. Here it should be pointed out that this behavior of the soluticn cannot be considered to be instability, since we can strictly prove stability in (2.5).

Based on the above discussion and the analysis of the acoustic equations made in [18], we can draw the following conclusions.

1. From the standpoint of stability, the best system is the first-order system with linear interpolaticn - system 1. This system is monotonic and contains the dissipative terms necessary for finding

solutions which are not very regular.

2. System II can cause unfavorable fluctuations for irregular solutions, but nevertheless makes it possible to use larger intervals. Therefore, it is advisable to use this system for numerically determining rather regular solutions.

3. It is expedient to write a single computer program for the net-characteristic method, and select the order of interpolation on the basis of the nature of the problem.

4. Using quadratic (or higher-order) interpolation at $\nu \neq 1/2$ makes the difference system unstable at a_i = const and stable only at $a_i^2 = 0(\tau)$. Checking for many examples of hyperbolic equations showed that the first-order systems for t and higher-order systems for x_1, x_2 are only stable at $a_i^2 = 0(\tau)$.

§3. Divergent Systems of the Net-Characteristic Method

The concept of the net-characteristic method can also be used to construct divergent difference systems. First we will consider a system of hyperbolic equations in divergent form with two independent variables

$$U_t + F_x = f \qquad F = F(t, x, u), \ f = f(t, x, u) \qquad (3.1)$$

Suppose a matrix with fixed components M, assigned in space U, has the proper values μ_i (i = 1, 2, ..., N) with the same sign as the proper values of matrix F_u, i.e., they have the same diagonal matrix $L = \{sign \mu_1, sign \mu_2, ..., sign \mu_n\}$. Multiplying (3.1) by linearily independent latent vectors ω_i of matrix M in scalar form, we will have

$$\omega_i U_t + \omega_i F_x = \omega_i f \qquad (i = 1, 2, ... N) \qquad (3.2)$$

Based on the independence of ω_i, this system is equivalent to the initial system (3.1). Approximating (3.2) with consideration of the characteristic directions of matrix M by the one-sided differences relative to x, we will have

$$\omega_i \frac{U_m^{n+1} - U_m^n}{\tau} + \omega_i \begin{cases} \dfrac{F_{m+1}^n - F_m^n}{h} & \text{at} \quad \mu_i < 0 \\[2mm] \dfrac{F_m^n - F_{m-1}^n}{h} & \text{at} \quad \mu_i > 0 \end{cases} = \omega_i f$$

$$\omega_i \frac{U_m^{n+1} - U_m^n}{\tau} + \frac{\omega_i}{h} \left[\Delta_m F - sign \mu_i \Delta_m^2 F \right] = \omega_i f$$

Multiplying the difference equations on the left written in matrix form by Ω^{-1} and using the previously introduced designations, we will

obtain the first-order divergent system

$$U_m^{n+1} = U_m^n - \delta \Delta_m F + \sigma \Omega^{-1} L \, \Omega \Delta_m^2 F + \tau f_m^n \qquad (3.3)$$

We can consider the difference system for version U in order to study the local stability of (3.3), and the system will be stable, e.g., at $M = A = F_u$. During practical calculations, for each point $(n + 1, m)$ we can use $M = A_m^n$ or retain it for other m when the proper values of F_u do not change sign.

For systems of hyperbolic equations with a large number of variables

$$U_t + (F_1)_{x_1} + (F_2)_{x_2} + \cdots = f \qquad (3.4)$$

the first-order divergent system can be written analogously to (3.3) and (1.8):

$$U_{m\ell}^{n+1} = U_{m\ell}^{n} - \sigma_1 \Delta_m F_1 + \sigma_1 \Omega_1^{-1} L_1 \Omega_1 \Delta_m^2 F_1 -$$

$$- \sigma_2 \Delta_\ell F_2 + \sigma_2 \Omega_2^{-1} L_2 \Omega \Delta_\ell^2 F_2 +$$

$$+ \tau f_m^n + 0(\tau^2 + h_1^2 + h_2^2) \qquad (3.5)$$

We will point out that the simplest form of divergent systems is used when solving problems in the regions in which the proper values of the matrices $(F_1)_u$, $(F_2)_u$, ... do not change sign. Then it suffices to calculate the matrices in (3.3) and (3.5) at one point.

We will compare the divergent system of the net-characteristic method with the Laks system. For equation (3.1), the Laks system is:

$$U_m^{n+1} = \frac{U_{m+1}^n + U_{m-1}^n}{2} - \sigma \Delta_m F = U_m^n - \sigma \Delta_m F + \Delta_m^2 U + \tau f_m^n \qquad (3.6)$$

This system is stable when the CFL condition is satisfied, but unlike (3.3), it is not flexible, i.e., for small ϵ, which is unavoidable with nonlinear equations due to the selection of the minimum interval using the CFL condition, it approximates a parabolic equation (see

[15]). The absence of σ in the second difference in (3.6) also causes the branches to be "spread" into a larger number of steps than in system (3.3).

In conclusion, we will give the second-order divergent system for equation (3.4) without deriving it:

$$U_{m\ell}^{n+1} = U_{m\ell}^{n} - \sigma_1 \Delta_m F_1 - \sigma_2 \Delta_\ell F_2 + \sigma_1^2 A_1 \Delta_m^2 F_1 + \sigma_2^2 A_2 \Delta_\ell^2 F_2 +$$

$$+ \tau\left[f_m^n + \frac{\tau}{2}\varphi_m^n\right] + O(\tau^3 + h_1^3 + h_2^3)$$

$$A_1 = (F_1)_u \quad , \quad A_2 = (F_2)_u$$

$$\varphi = A_1(F_2)_{x_1 x_2} + A_2(F_1)_{x_1 x_2} - \left[(A_1)_{x_1} + (A_2)_{x_2}\right]\left[f - (F_1)_{x_1} - (F_2)_{x_2}\right] +$$

$$+ f_t - A_1 f_{x_1} - A_2 f_{x_2}$$

$$(3.7)$$

The values of A_1, A_2 and φ are calculated at point (n, m, ℓ) with precision down to the terms on the order of $O(h_1 + h_2)$.

§4. Using the Net-Characteristic Method to Calculate Multidimensional

Gas Flows

The above method can be used to study the movement of an inviscid non-heat-conductive gas in the case of purely supersonic steady flows, as well as in the general case of unstationary gas-dynamic equations. The corresponding difference equations of the net-characteristic (NC) [CX] method are partially given in [18], [19], while the method of calculating the boundary points (on the surface of the body and on the shock wave) is described in [8], [15]. Only some of the most typical examples of the numerical calculations are given below.

The main properties of the NC method were studied in the problem of supersonic steady flow on the side surface of blunt cones. Many results were gathered for this problem, and different numerical methods were obtained [20]. The calculations showed that the parameters on the body and the shock wave are computed with an error of 1-2o/o both using system I, and using system II of the NC method. But here system I requires approximately four times as many nodes as system II for obtaining the same degree of accuracy. But the second-order system results in fluctuations ("ripples") around a certain averaged curve which are especially noticeable when calculating the profiles of gas-dynamic functions in regions with a marked change in parameters and with discontinuities in the

derivatives. The pattern shown in Fig. 1 is the most characteristic.
This figure shows the pressure profiles P on different meridional
planes φ = const at a distance of five radii of blunting for a
spherically blunted cone with a half-angle of aperture of 10°
streamlined by a supersonic gas flow with an adiabatic index of γ =
1.4 at a Mach number of M = 6 and an angle of attack of α = 5°. The
solid lines show the results of the calculation using the direct
system of the characteristic method [21], while the crosses and
circles indicate the data calculated using systems I and II of the NC
method. It is evident that while method [21] clearly isolates the
discontinuities (they originate due to the interruption of curvature
at the point of contact of the cone with the sphere), the NC method
smoothes these discontinuities, whereupon system I is regular, and
system II has marked fluctuations.

Fig. 1.
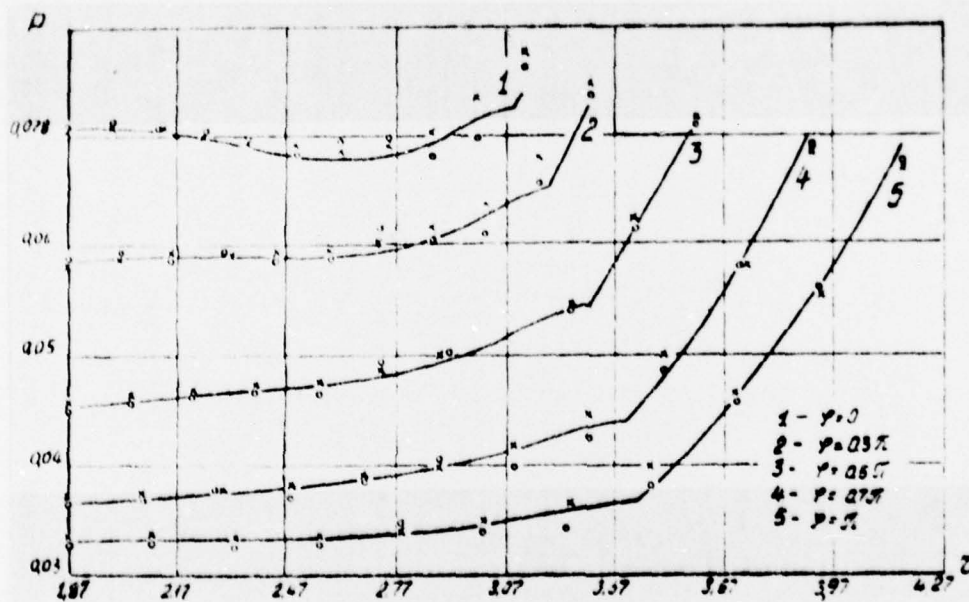
A similar pattern is also observed when performing a "through" calculation of the zones of discontinuities in the functions. In order to test the possibility of applying the NC method to the through calculation of discontinuous solutions, various problems were considered. Figure 2 shows the flow pattern about a two-degree cone at $M = 3$ and $\gamma = 1.4$, as well as the pressure distribution P on the body. The internal connections of the shocks were found by the through calculation using system I. The solid lines show the results from [22]. Figure 3 shows the flow pattern about a plane body and the pressure distribution on the body at $M = 5$. Here the leading shock wave 1 was isolated by calculation, while the "suspended" shock wave 2 was obtained by the through calculation. The upper curve 2 depicts the precise position of the shock.
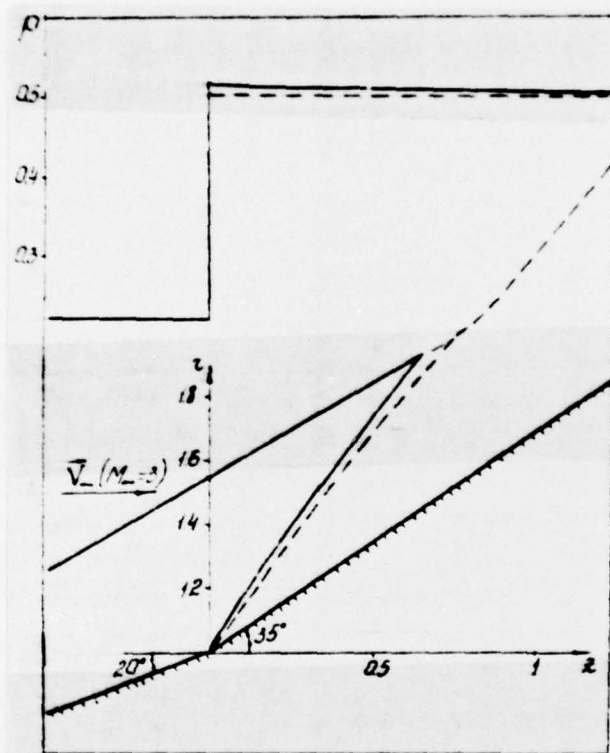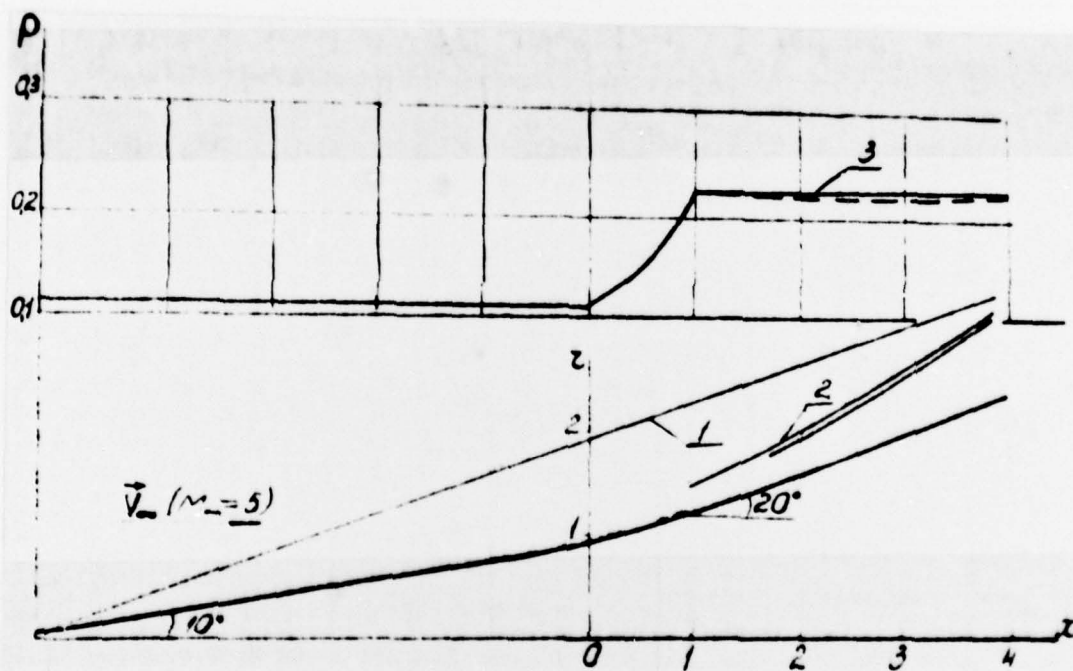
Fig. 2.

Fig. 3.

The following should be pointed out regarding the use of the NC method for a through calculation. The calculations showed that discontinuities which are not very marked can be obtained using system I of the NC method. This system is not divergent, but in the difference system, the pressure and angles of slope of the velocity vector for steady flows [18] are determined independent of the energy equation. Here the coefficients in these equations depend mainly on the local Mach number, and vary little in many problems. The energy equation is used in a such a form that the dissipative term in the presence of a shock wave results in the Hugoniot condition for the relationship between pressure and enthalpy. If it is necessary to cross marked discontinuities, e.g., a leading shock wave, by a through calculation, it is necessary to use the divergent system of the NC method given in §3 to increase accuracy.

As we know, many problems of gas dynamics require the simultaneous calculation of the regions of subsonic and supersonic flow. In order to make it possible to apply the NC method to these problems, we can introduce a new independent variable t such that the equation becomes hyperbolic everywhere and the solution at $t \to \infty$ gives us the solution of the initial equations. This is done simply by retaining, e.g., physical time t, for gas-dynamic equations. This is

precisely how problems of three-dimensional flow about segmented

conical bodies and the direct problem of flow in a nozzle were solved

by the net-characteristics method.

Figure 4 gives the flow pattern for a segmented conical body

shaped like the Apollo launch capsule, which is streamlined at an

angle of attack of $\alpha = 0$. The shock waves and sound lines (a) are

shown, along with the pressure distribution on the body (b) for a

perfect gas at Mach numbers of the incoming flow of $M = 2$, 6 and 20

and adiabatic indices $\gamma = 1.4$, as well as with consideration of the

equilibrium physico-chemical processes in the air at velocities of

the incoming flow of $V_\infty = 5$ km/s ($M = 16.1$) and $V_\infty = 7.5$ km/s ($M =$

23.5). Similar results are shown in Fig. 5 at $M = 6$ and $\gamma = 1.4$ for

three-dimensional streamlining at angles of attack of $\alpha = 10°$, 20°
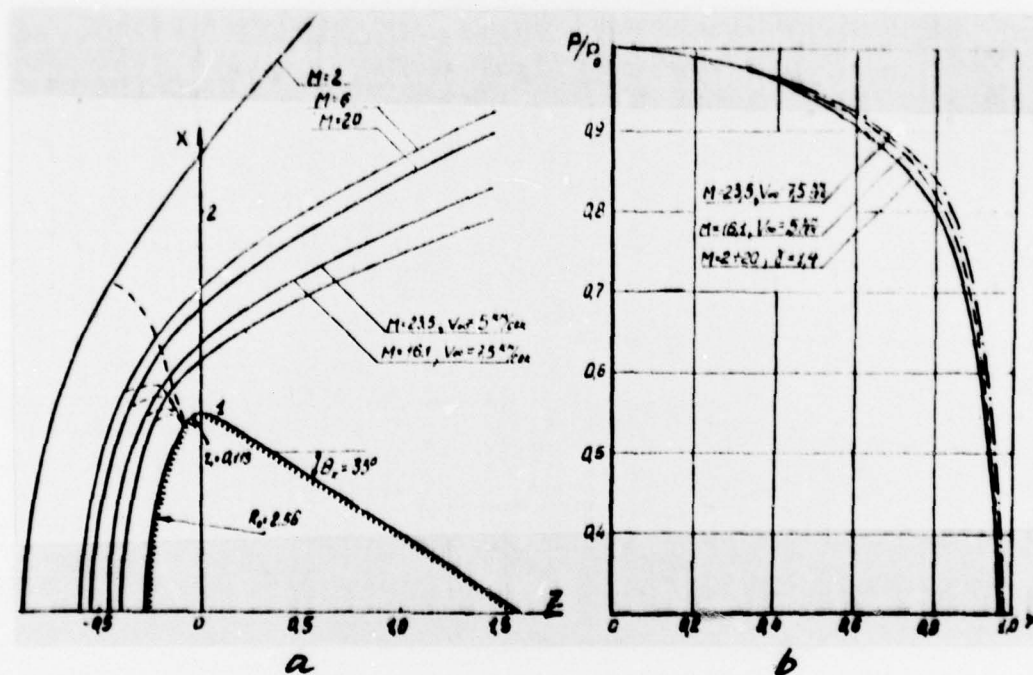
and 25°.
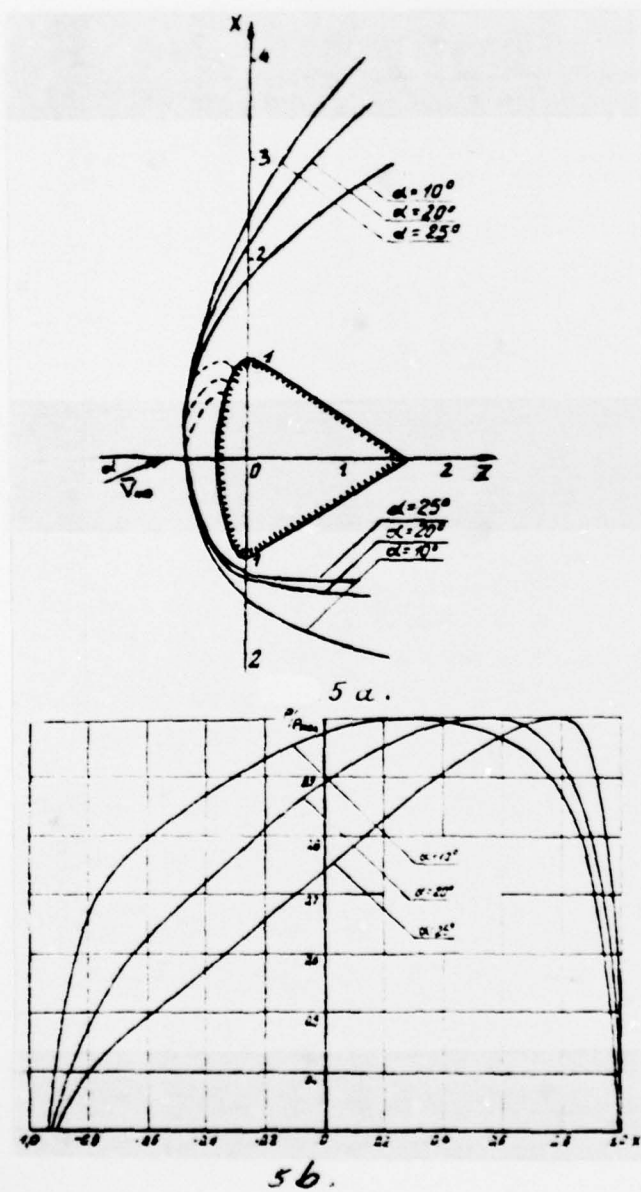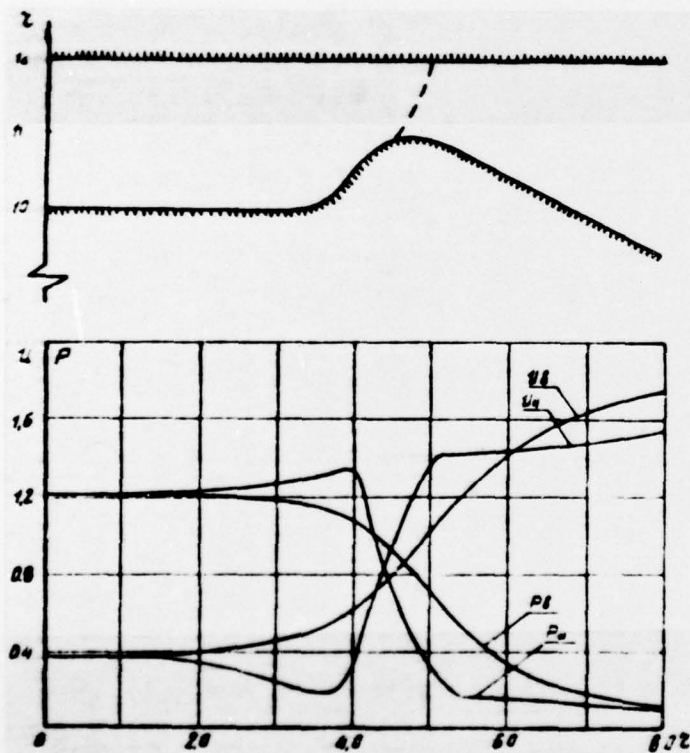
Fig. 4.

Fig. 5.

5 a.

5 b.

Figure 6 shows another problem - the calculation of a circular nozzle with an assigned shape. It gives the shape of the nozzle, the sound line, and the pressure distribution and longitudinal velocity on the upper and lower walls.

Fig. 6.

Conclusion

The net-characteristic method proposed for solving
multidimensional hyperbolic equations makes it possible to calculate
different gas-dynamics problems. This report focused mainly on the
properties and peculiarities of the difference system, and the
results given are illustrative, but they nevertheless make it
possible to determine the class of problems which the system can be
used to solve. The analysis and numerous experimental calculations
show that the method is stable and monotonic, and that it makes it
possible to perform "through" calculations which do not generate
peculiarities.

This method was developed at the Moscow Physico-Technical
Institute as part of the numerical method research plan conducted
under the guidance of Professor O. M. Belotserkovskiy. The main
concepts of the method are published in the report by K. M. Magomedov
and A. S. Kholodov [18]. A. S. Kholodov, V. I. Kosarev and V. V.
Pirogov conducted many of calculations using the net-characteristic
method.

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